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Turning bosons into fermions: exclusion statistics, fractional statistics and the simple harmonic oscillator

A D Speliotopoulos†
Institute of Physics, Academia Sinica, Nankang, Taipei, Taiwan

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Abstract. Motivated by Haldane's exclusion statistics, we construct creation and annihilation operators for g-ons using a bosonic algebra. We find that g-ons appear due to the breaking of a discrete symmetry of the original bosonic system. This symmetry is intimately related to the braid group and we demonstrate a link between exclusion statistics and fractional statistics.

In 1991 Haldane proposed that excitations which obey fractional statistics may exist in certain condensed matter systems even in dimensions other than two [1]. This was based on his observation that such excitations are present in the Calogero–Sutherland model [2–4]. For these excitations, which were dubbed 'g-ons' by Nayak and Wilczek [5], he proposed that the number of available one-particle states changes with increasing occupation of the state. Namely, if d is the number of one-particle states available for the particle and N is its occupation number, then

$$\Delta d = -g\Delta N \tag{1}$$

where g determines the statistics of the particle. Since the number of one-particle states for bosons does not change with N, $g = 0$ for bosons. For fermions, on the other hand, the number of available one-particle states decreases by one with the addition of even one fermion, giving a $g = 1$. This definition of statistics has the advantage of making no specific reference to the dimensionality of the system which is being studied while the usual notion of fractional statistics is often made in reference to anyons [6] which traditionally have only been present in two spatial dimensions. We also note that excitations which have 'fractional charge' have had a long history in both high energy physics and condensed matter physics. An explicit example of such excitations can be found in the massive Thirring model in two dimensions [7]. Moreover, the notion of particles having fractional charge and obeying parastatistics have been postulated in high energy physics in connection with the quark model of hadrons since the 1960s (see [8] for a thorough review).

Since Haldane's work there have been a number of papers advancing his ideas [5, 9–15] using thermodynamic arguments. This was done primarily by using the second of Haldane's proposals: that the dimensionality of the Hilbert space of N g-ons is

$$D = \frac{[d + (1-g)(N-1)]!}{N![d-1-g(N-1)]!}. \tag{2}$$

† Current address: Higher Dimension Research, Inc., 7650 Currell Blvd. Suite 340, St. Paul, MN 55125, USA. E-mail address: ads@hdri.com

Applying the ground breaking work of Yang and Yang [16] who first noticed that the number of available one-particle states can differ depending on dynamics, equation (2) was used to construct a partition function for g-ons. (See also [17] for a review of this technique as applied to Haldane's fractional statistics.) This was then compared with the partition function of various known systems. Using this argument Murthy and Shanker [11] were able to show that anyons also obey exclusion statistics.

In another approach, Karabali and Nair [18] have recently attempted to use operator methods to realize exclusion statistics algebraically (see also [19] and [20]). Specifically, they proposed the existence of creation and annihilation operators $\tilde{a}$ and $\tilde{a}^\dagger$ for g-ons by requiring that $\tilde{a}^{m+1} = 0$. The resultant Hilbert space is therefore finite-dimensional containing at most m possible occupation states. As the system looses a one-particle Hilbert space as soon as the occupation number for the state changes by m, $g = 1/m$. Although $m(1/g)$ is necessarily an integer, since Nayak and Wilczek have shown that the thermodynamical distribution functions for g and $1/g$ are related by a duality property, this limitation does not seem to be to troublesome. Of greater concern is that since a specific representation of these operators was not known, the resulting commutation relations for $\tilde{a}$ and $\tilde{a}^\dagger$ could not be determined uniquely.

In this paper we shall extend Karabali and Nair's analysis by constructing explicitly the creation and annihilation operators for g-ons using a bosonic algebra. Since a $(1/2)$-on is a fermion, as a byproduct of our analysis we have a procedure for changing a boson into a fermion (or in general a g-on). Turning bosons into fermions or fermions into bosons, the process of bosonization, is well-known in field theory ([21–24]; see also [25]). Instead of using the normal field theoretic methods, however, we use the bosonic number operator to define an operator whose eigenvalues are abelian representations of the braid group, commonly identified as the 'statistics phases' for anyons [6]. Projection operators are then constructed which project states of the bosonic Hilbert space $\mathcal{H}$ into states with definite statistics phase. These projection operators are then used to define the g-on creation and annihilation operators. We shall further see from this construction that g-ons appear in the bosonic system as the result of the breaking of a discrete symmetry of the original bosonic Hilbert space. In the spirit of Haldane's original work, this analysis is done *without* reference to any specific spacetime dimension.

Some of the techniques of this paper have previously been applied in various forms to different systems. Brandt and Greenberg [26] have used a similar technique to construct generalized bose operators which change the occupation number of a state by any positive integer. Agranovich and Toshich [27] used a similar method to construct creation and annihilation operators for Paulions, which was later generalized by Ilinskaia and Ilinski [28].

Denoting the usual bosonic operators by a and $a^\dagger$ and their number operator by $N = a^\dagger a$, we begin with the unitary operator

$$B_m \equiv \exp\left(\frac{2\pi\mathrm{i}}{m+1}N\right) \tag{3}$$

where m is a non-negative integer. One can easily show that

$$B_m a B_m^\dagger = \mathrm{e}^{-2\pi\mathrm{i}/(m+1)}a \qquad \mathcal{B}_m a^\dagger B_m^\dagger = \mathrm{e}^{2\pi\mathrm{i}/(m+1)}a^\dagger. \tag{4}$$

Consequently, $[a, B_0] = 0 = [a^\dagger, B_0]$, and B_0 commutes with every operator in the algebra generated by a and $a^\dagger$. It is therefore (a Casimir operator) proportional to the identity operator I. Since the eigenvalues of N are the non-negative integers, this proportionality constant is simply unity. This is a special case of the more general result: if $f(x)$ is a periodic, analytic function, then $[a, f(N)] = 0$ and $f(N) = f(0)\mathrm{I}$.

Now consider B_m for $m > 0$ which has eigenvalues

$$e^{2\pi i j/(m+1)} \tag{5}$$

that are the $m+1$ roots of unity. Since only the ratio $j/(m+1)$ matters, it is understood from now on that $0 \leqslant j \leqslant m$. It is also known that for a fixed value of $j/(m+1)$, equation (5) is a one-dimensional (abelian) representation of the braid group $e^{-i\pi\nu}$ if one identifies $\nu = -2j/(m+1)$ [6] (up to an even integer). In this context, equation (5) is often also called the 'statistics phase' of an anyon. Consequently, we shall call B_m the quantum *braid* operator.

Next, $B_m|j+(m+1)q\rangle = e^{2\pi i j/(m+1)}|j+(m+1)q\rangle$, where $|n\rangle$ is a state in $\mathcal{H}$ and $q \geqslant 0$ is an arbitrary integer. The bosonic Hilbert space $\mathcal{H}$ is also spanned by eigenstates of B_m. These states can be isolated by using the projection operators

$$P_j^m = \frac{1}{m+1}\sum_{k=0}^{m} \exp\left(\frac{2\pi i k}{m+1}(N-j)\right) \tag{6}$$

which have the properties

$$P_j^{m\dagger} = P_j^m \qquad P_j^m P_k^m = P_j^m \delta_{j,k} \qquad \sum_{j=0}^{m} P_j^m = \mathrm{I} \tag{7}$$

and $P^m_{j\pm(m+1)} = P_j^m$. It is easily seen that $P_j^m|n+(m+1)q\rangle = \delta_{j,n}|n+(m+1)q\rangle$ and the original Hilbert space decomposes into $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_m$. Each state in $\mathcal{H}_j$ is an eigenstate of B_m with eigenvalue (5) and has a definite statistics phase.

We then use P_j^m to form the composite operators

$$e_j^m \equiv a P_j^m \qquad e_j^{m\dagger} \equiv P_j^m a^\dagger \tag{8}$$

where $0 \leqslant j \leqslant m$. Using equation (4),

$$P_j^m a = a P_{j+1}^m \qquad P_j^m a^\dagger = a^\dagger P_{j-1}^m \tag{9}$$

and we find that

$$e_j^m e_k^m = e_j^m e_{j+1}^m \delta_{k,j+1} \qquad e_k^{m\dagger} e_j^{m\dagger} = e_{j+1}^{m\dagger} e_j^{m\dagger} \delta_{k,j+1} \qquad \{e_j^m, e_k^{m\dagger}\} = T_j^m \delta_{j,k}$$
$$[T_j^m, T_k^m] = 0 \tag{10}$$

while

$$[e_j^m, T_{j+1}^m] = e_j^m + e_j^m N_j^e \qquad [e_j^m, T_{j-1}^m] = -N_{j-1}^e e_j^m \tag{11}$$

and $[e_j^m, T_k^m] = 0$ for $|j-k| \neq 1$. $N_j^e \equiv e_j^{m\dagger} e_j^m$, $N = \sum_j N_j^e$ and $[N_j^e, T_k^m] = 0$. Written in this way the original background bosonic operators do not explicitly appear, the algebra closes and no other operators need to be introduced. These e_j^m will be used to construct the creation and annihilation operators for g-ons and their resultant Hilbert spaces. To demonstrate that the commutation relations (10) are sufficient to determine the g-on Hilbert space up to an overall constant, we shall make no further reference to the underlying bosonic algebra.

In general, the creation and annihilation operators a_j^m and $a_j^{m\dagger}$ for $(g = 1/m)$-ons are defined by

$$a_j^m = \sum_{k=1}^{m} e_{j+k}^m \tag{12}$$

and we see that $(a_j^m)^{m+1} = 0$. The $(1/m)$-on Hilbert space is m-dimensional and spanned by the states

$$|p\rangle_j^m = \frac{(a_j^{m\dagger})^p|\Omega\rangle_j^m}{\sqrt{\lambda_{j+1}^m(\lambda_{j+1}^m+1)\ldots(\lambda_{j+1}^m+p-1)}} \tag{13}$$

for $1 \leqslant p \leqslant m$ with the ground state being $|\Omega\rangle_j^m$. They are eigenstates of the $(1/m)$-on number operator N_j^m with eigenvalues $\lambda_{j+1}^m + p$, as well as of $T_{j+1}^m|p\rangle_j^m = \lambda_{j+1}^m\delta_{p,1}|p\rangle_j^m$. Once again $\lambda_{j+1}^m > 0$ is an arbitrary constant which, because $T_{j+1}^m|\Omega\rangle_j^m = \lambda_{j+1}^m|\Omega\rangle_j^m$, ultimately depends on the ground state of the system. For the underlying bosonic Hilbert space, $\lambda_{j+1}^m = j + 1 + (m+1)q$.

For $m = 1$, we have only the one operator $a_0^1 \equiv e_1^1$. Then from equation (10), $(a_0^1)^2 = 0$ and $\{a_0^1, a_0^{1\dagger}\} = T_1^1$. Since, however, $[a_0^1, T_1^1] = 0$, if we restrict ourselves to only those operators and Hilbert space generated by a_0^1 and $a_0^{1\dagger}$, then T_1^1 is once again a multiple of the identity which can be set to unity by re-scaling a_0^1. a_0^1 and $a_0^{1\dagger}$ therefore obey the usual *fermionic anticommutation* relations and generate a fermionic Hilbert space.

Even though we still have the operator e_0^1, no other interesting operator can be constructed for $m = 1$ aside for the trivial replacement of $e_0^1 \leftrightarrow e_1^1$. Once e_0^1 is also included in the sub-algebra the complete bosonic Hilbert space will be reconstructed as can be seen from the completeness relation in equation (7). We will no longer be projecting H into a sub-Hilbert space with a definite statistics phase. This is a reflection of the very well known result that for $m = 1$ the statistics phase is ± 1 and only fermions or bosons can be present.

For $m = 2$ we construct the creation and annihilation operators for $(1/2)$-ons by taking, for an arbitrary but fixed j,

$$a_j^2 = e_{j+1}^2 + e_{j+2}^2. \tag{14}$$

$(a_j^2)^3 = 0$, as can easily be seen from the commutation relations. To construct the Hilbert space, we take $|\Omega\rangle_j^2$ as the ground state. It is an eigenstate of both T_{j+1}^2 and T_{j+2}^2 with eigenvalues λ_{j+1}^2 and λ_{j+2}^2, respectively. This is possible because both these operators commute among themselves as well as the $(1/2)$-on number operator $N_j^2 \equiv a_j^{2\dagger}a_j^2$. Then, because $e_{j+2}^2e_{j+1}^2 = 0$, one can show that $\lambda_{j+2}^2 = 0$. The $(1/2)$-on Hilbert space is then spanned by the normalized states

$$|\Omega\rangle_j^2 \qquad \frac{a_j^{2\dagger}}{\sqrt{\lambda_{j+1}^2}}|\Omega\rangle_j^2 \qquad \frac{(a_j^{2\dagger})^2|\Omega\rangle_j^2}{\sqrt{\lambda_{j+1}^2(\lambda_{j+1}^2+1)}}. \tag{15}$$

They are eigenstates of $N_j^2 = a_j^{2\dagger}a_j^2$ with eigenvalues 0, λ_{j+1}^2 and $\lambda_{j+1}^2 + 1$, respectively, and of T_{j+1}^2 with eigenvalues $\lambda_{j+1}^2, \lambda_{j+1}^2$, and 0. The constant $\lambda_{j+1}^2 > 0$ itself cannot be determined by equation (10) alone. However, using the underlying bosonic Hilbert space we find that $\lambda_{j+1}^2 = j + 1 + 3q$.

In the $m \to \infty$ limit, equation (12) becomes an infinite sum, and we find that $(a_j^\infty)^l \neq 0$ for any finite l. Using standard arguments we can show that $\lambda_{j+1}^\infty = 1$ and we simply recover the usual bosonic algebra and Hilbert space. This can also be seen heuristically from equation (5), by looking at the spectrum of B_m for finite m and taking $m \to \infty$. We therefore identify $B_\infty = \mathrm{I}$ and note that $B_0 = B_\infty$.

$B_m|p\rangle^m = \mathrm{e}^{2\pi\mathrm{i}p/(m+1)}|p\rangle^m$ for a B_m invariant ground state. Each $(1/m)$-on occupation state is an eigenstate of the braid operator with the state containing p of the $(1/m)$-ons having a statistics phase $e^{2\pi i p/(m+1)}$. Each $(1/m)$-on thereby has a statistics phase of $e^{2\pi i/(m+1)}$. For $m = 1$ this is just -1, as expected for fermions, while for $m \to \infty$ it is 1, as expected for bosons.

We can understand the physics behind this construction of g-ons by using the following symmetry arguments. We first introduce the notion of B_m-parity, which like the usual parity $\mathcal{P}$ is a discrete symmetry and is generated by B_m. Since, however, $B_m^{m+1} = \mathrm{I}$, the eigenvalues of B_m are in general complex while $\mathcal{P}$ is a $\mathbb{Z}_2$ symmetry. The original bosonic system is, of course, B_m-parity invariant and $\mathcal{H}$ itself is spanned by states with definite B_m-parity. In the construction of the g-on operators a_j^m, $a_j^{m\dagger}$, however, only the projection operators $P_{j+1}^m, \ldots, P_{j+m}^m$ were used. P_j^m itself was purposely left out. In effect, a_j^m represents the projection of any bosonic state into a m-dimensional subspace spanned by these projection operators after which a is applied. B_m-parity is explicitly broken by hand and the states $|j + (m+1)q\rangle$ that P_j^m projects into forms the ground state $|\Omega\rangle_j^m$ for the g-on Hilbert space. The choice of ground state for the g-ons is not unique and there are an infinite number of ground states which can be used to generate the Hilbert space, each corresponding to a different λ_{j+1}^m. These ground states are *not* equivalent, however, since the eigenvalues of N_j^m are λ_{j+1}^m are dependent.

Since the eigenvalues of B_m are in general complex, B_m is not a physical observable for $m > 1$. Its effect on the bosonic Hilbert space is nevertheless dramatic and observable. The breaking of this discrete symmetry reduces the infinite-dimensional bosonic Hilbert space into the finite-dimensional g-on Hilbert space. Indeed, the presence of g-ons in the system occurs precisely because this discrete symmetry is broken in the original bosonic system.

As an explicit example of this, consider the case $m = 1$, for which $B_1 = (-1)^N$, $(B_1)^2 = \mathrm{I}$ and is a $\mathbb{Z}_2$ symmetry. With respect to this operator, $\mathcal{H}$ consists of both even ($|2n\rangle$) and odd ($|2n+1\rangle$) states. The fermionic operators a_0^1 are, however, constructed from P_1^1 only. P_0^1 was not, and could not, be used or else the original Hilbert space would be reproduced. B_1-parity is explicitly broken and we find that the one fermion state $(a_0^1)^\dagger|\Omega\rangle_0^1$ is odd under B_1. It has statistics phase of -1, as expected. Also, if we identify $\mathcal{H}$ with the one-dimensional simple harmonic oscillator, then B_1^1 also functions as the usual parity operator. Equivalently, fermions appear in the bosonic system due to parity being broken. This breaking of parity is a well known effect for anyons. Unfortunately, due to its complex eigenvalues B_m does not have a physical interpretation for $m > 1$.

To conclude, we have constructed the creation and annihilation operators for g-ons; particles which obey Haldane's exclusion statistics. This was done using the usual bosonic creation and annihilation operators without any reference to a specific spacetime dimension. Physically, g-ons appear in the bosonic system as the result of the breaking of a discrete symmetry. Moreover, as the construction explicitly used the braid operator B_m whose eigenvalues consist of abelian representations of a braid group, we have established a link between Haldane's exclusion statistics, fractional statistics, the braid group and anyons. Indeed, taking $g = 1/m$ we have found that g-ons have a statistics phase of $e^{2\pi i g/(g+1)}$, and have finite-dimensional Hilbert spaces, precisely as one would expect for anyons. Consequently, denoting the usual statistics phase of an anyon by $e^{\pi i\alpha}$, we can identify $\alpha(g) = 2g/(g+1) + 2q$, where q is any integer. With the appropriate choice of q, we can always reduce α to lie within $0 \leqslant \alpha \leqslant 0$. With this restriction,

$$\alpha(g) = 2g/(g+1) \tag{16}$$

which has the correct limiting values at $g = 0, 1$ for bosons and fermions, respectively. Because in our approach $g = 1/m, 0 \leqslant g \leqslant 1$ and from equation (16) we find that $0 \leqslant \alpha \leqslant 1$.

Murthy and Shanker [11] have also shown that anyons obey exclusion statistics. In their analysis a partition function for g-ons was constructed using equation (2) which was generalized to infinite-dimensional Hilbert spaces. A virial expansion is then performed on this partition function and g is shown to be very simply related to the second virial coefficient in the high temperature limit. Since the second virial coefficient has been calculated for the anyon gas [29], they find that

$$g_{ms} = \alpha_{ms}(2 - \alpha_{ms}). \tag{17}$$

Inverting this equation, one finds that $\alpha_{ms} = 1 \pm \sqrt{1 - g_{ms}}$. For α_{ms} to be real, $0 \leqslant g_{ms} \leqslant 1$ and in this range, either $0 \leqslant \alpha_{ms} \leqslant 1$ or $1 \leqslant \alpha_{ms} \leqslant 2$. Although the range of both g_{ms} and α_{ms} are similar to our result (16), equation (17) is different from our result. Their result was obtained via a virial expansion and, as they have pointed out, is valid for a general anyon gas only if *all* the virial coefficients are finite for the anyon gas, a result which is not yet known. Our results would seem to suggest that either these virial coefficients are not finite, or else the relationship they derived is valid only in the high temperature limit near $\alpha_{ms} = 0, 1$ when $\alpha \approx \alpha_{ms}$.

Traditionally, anyons have been associated with two dimensions where the homotopy class $\pi_1(M_n)$ on the configuration space M_n of n hard core particles is non-trivial. The intertwining worldlines of these particles in the Feynman path integral formalism form a representation of the abelian braid group. By using operator instead of path integral methods to realize the braid group we have extended the notion of anyons to arbitrary dimensions. There is, however, a fundamental difference in the two approaches. In our approach g-ons appear because an underlying discrete symmetry of the bosonic Hilbert space is *broken*, while in the standard description anyons are present precisely because the braid group is a fundamental symmetry group which is *not* broken. This may also be the cause of the differences between our $\alpha(g)$ and $\alpha_{ms}(g)$. It would also be interesting to see if this symmetry breaking can occur dynamically instead of by hand as we have done.

References

[1] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[2] Ha Z N C 1995 *Phys. Rev. Lett.* **73** 1574
[3] Bernard D *Preprint* SPhT-94-155 hep-th/9311013
[4] Minahan J A and Polychronakos A P 1994 *Phys. Rev.* B **50** 4236
[5] Nayak C and Wilczek F 1994 *Phys. Rev. Lett.* **73** 2740
[6] Wilczek F (ed) 1990 *Fractional Statistics and Anyon Superconductivity* (Singapore: World Scientific)
[7] Korepin V E 1979 *Theo. Math. Phys.* **41** 953
[8] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (New York: Springer)
[9] Wu Y S 1994 *Phys. Rev. Lett.* **73** 922
[10] Bernard D and Wu Y S *Preprint* SPhT-94-043, UU-HEP/94-03 cond-mat/9404025
[11] Murthy M V N and Shanker R 1994 *Phys. Rev. Lett.* **72** 3629
[12] Fukui T and Kawakami N 1995 *J. Phys. A: Math. Gen.* **28** 6027–38
[13] Mitra P *Preprint* hep-th/9411236
[14] Bernard D *Preprint* hep-th/9311013
[15] Fukui T and Kawakami N 1997 *Nucl. Phys.* B **483** 663–80
[16] Yang C N and Yang C P 1969 *J. Math. Phys.* **10** 1115
[17] Bernard D *Preprint* SPhT-94-119, hep-th/9411017 (Presented at NATO Advanced Study Institute, Les Houches Summer School Session 62: Fluctuating Geometries in Statistical Mechanics and Field Theory, August 1994)
[18] Karabali D and Nair V P 1995 *Nucl. Phys.* B **438** 551–60
[19] Ilinski K N and Gunn J M F *Preprint* hep-th/9503233
[20] Ilinshaia A V, Ilinski K N and Gunn J M F 1996 *Nucl. Phys.* B **458** 562
[21] Coleman S 1975 *Phys. Rev.* D **11** 2088

[22] Mandelstam S 1975 *Phys. Rev.* D **11** 3026
[23] Halpern M B 1975 *Phys. Rev.* D **12** 1684
[24] Halpern M B 1975 *Phys. Rev.* D **13** 337
[25] Stone M 1994 *Bosonization* (Singapore: World Scientific)
[26] Brandt R A and Greenberg O W 1969 *J. Math. Phys.* **10** 1168
[27] Agranovich V M and Toshich B S 1968 *Sov. Phys.–JETP* **26** 104
[28] Ilinskaia A V and Ilinski K N 1996 *J. Phys. A: Math. Gen.* **29** L23
[29] Arovas D P, Schrieffer J R, Wilczek F and Zee A 1985 *Nucl. Phys.* B **251** 117
Dowker J S 1985 *J. Phys. A: Math. Gen.* **18** 3521
[30] Wu Y S 1984 *Phys. Rev. Lett.* **52** 2103